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Singular Perturbations of Ordinary Differential Equations

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SINGULAR PERTURBATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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SINGULAR PERTURBATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

Introduction

Since so few differential equations can be integrated exactly, it is important to develop approximate methods for solving such equations. One important approximation method, and almost the only one, is to express the desired solution of a differential equation as a modification or perturbation of the solution of a simpler, that is, more nearly solvable, equation. This method is known as the method of perturbations and its validity usually depends on the fact that the modification in the equation or in the solution is small in some sense. Frequently, the smallness of the perturbation depends on the size of some parameter in the equation and we may be able to express the desired solution as a convergent power series in terms of the parameter.

We shall discuss the perturbation method for ordinary differential equations depending upon a small parameter ε. We begin in Chapter I with the regular case, that is, the case where naive methods give the desired solution and sometimes even give it as a convergent power series in the parameter. As illustrations of this procedure, we shall obtain the Neumann series and the Fredholm expansion for the solution of an integral equation. We shall also indicate how to obtain the solution of the eigenvalue problem for an ordinary differential equation as a power series in ε.

In Chapter II we begin the study of singular perturbation problems, that is, problems in which the standard methods fail and some special device has to be employed. We consider differential equations in which the coefficient of the highest derivative is multiplied by the small parameter ε; therefore, in the limit as ε approaches zero, the order of the differential equation is lowered and we have too many boundary conditions. The question of whether the limiting problem makes sense and if so, which, if any, of the given boundary conditions should be used will be discussed in a very general case. It is also shown that the boundary conditions which are lost give rise to a "boundary layer" effect.

In Chapter III we treat the relaxation oscillations for Van der Pol's equation with a large value of the parameter. The problem is to find the period of the oscillation in terms of the parameter. We shall obtain two terms in the expansion for the period.

Chapter I

REGULAR PERTURBATIONS

Iteration and the Neumann Series

Consider the problem of solving the equation

(1) $$Lu - \varepsilon Mu = f$$

where u is a given function and L and M are given operators. We assume that we know how to solve the equation

$$Lu = f$$

and wish to use the solution of this equation to solve (1). Suppose the inverse operator L^{-1} is known. Applying it to (1), we get

(2) $$u - \varepsilon Ku = g,$$

where we have put

$$L^{-1}M = K, \quad L^{-1}f = g.$$

We solve (2) by an iteration method. We write it as

(3) $$u = g + \varepsilon Ku$$

and then obtain a sequence of approximations $u_0, u_1, u_2, \ldots$ by putting

(4) $$u_{n+1} = g + \varepsilon Ku_n, \qquad n = 0,1,2,\ldots$$

$$u_0 = g.$$

There are now two problems. First, to show that the sequence u_n converges to a limit and secondly, to show that this limit satisfies (3).

The usual procedure for investigating the convergence of the sequence begins by obtaining an estimate for the difference of successive approximations. From (4) we have

(5) $$u_{n+1} - u_n = \varepsilon(Ku_n - Ku_{n-1}).$$

Note that we do not assume that K is a linear operator. Introduce some norm, that is, a measure for the closeness of two functions v and w and let us denote it by the symbol $\|v - w\|$. For example, one possibility is to take

$$\|v - w\| = \max|v(x) - w(x)|$$

for x in some interval of interest. Put

(6) $$\rho_{n+1} = \|u_{n+1} - u_n\|.$$

We suppose that the operator K is such that

(7) $$\|Ku_n - Ku_{n-1}\| \leq C\|u_n - u_{n-1}\| = C\rho_n.$$

Then, because of (5)

$$\rho_{n+1} \leq C\rho_n \leq C^2\rho_{n-1} \leq \cdots \leq C^n\rho_1.$$

Using this estimate, we see that the infinite series

(8) $$u_0 + (u_1 - u_0) + (u_2 - u_1) + \cdots$$

is majorized by the series

$$\|u_0\| + \rho_1 + \rho_2 + \cdots$$

or by

$$\|u_0\| + \rho_1 \sum_0^\infty C^n.$$

This series converges if $(C) < 1$ and therefore (8) also converges. But the n^{th} partial sum of (8) is u_n; therefore, the sequence u_n converges to a limit.

The proof that this limit satisfies equation (3) depends upon the operator K having some kind of continuity.

We shall not discuss this question further.

As an illustration of the method, we consider the problem of finding a solution of the non-linear ordinary differential equation

$$y' = f(x,y) \tag{9}$$

with $y(0) = y_0$. Note that we have omitted the parameter ε because it is not needed for convergence of the iteration method. Inverting (9), we get

$$y = y_0 + \int_0^x f[\xi, y(\xi)]d\xi .$$

Just as in (4), we set up an iteration procedure. We put $y_0(x) = y_0$ and

$$y_{n+1}(x) = y_0 + \int_0^x f[\xi, y_n(\xi)]d\xi . \tag{10}$$

For a norm we use

$$\rho_{n+1}(x) = \|y_{n+1} - y_n\| = |y_{n+1}(x) - y_n(x)| .$$

To prove the sequence $y_n(x)$ converges, we need some condition on $f(x,y)$. We assume that

$$\left|\frac{\partial f(x,y)}{\partial y}\right| < C$$

for all y and $0 \leq x \leq a$. Then

$$\begin{aligned} \left|f[\xi,y_n(\xi)] - f[\xi,y_{n-1}(\xi)]\right| &= \left|\int_{y_{n-1}(\xi)}^{y_n(\xi)} \frac{\partial f(\xi,\eta)}{\partial \eta} d\eta\right| \\ &\leq \left|\int_{y_{n-1}(\xi)}^{y_n(\xi)} C d\eta\right| \qquad (11) \\ &\leq C|y_n(\xi) - y_{n-1}(\xi)| = C\rho_n(\xi). \end{aligned}$$

From (10), we have

$$y_{n+1}(x) - y_n(x) = \int_0^x f[\xi, y_n(\xi)] - f[\xi, y_{n-1}(\xi)] \; d\xi$$

and using (11) we get

$$(12) \qquad \rho_{n+1}(x) \leq C \int_0^x \rho_n(\xi) d\xi .$$

The proof of convergence can be completed as before if we can obtain an estimate for $\rho_n(x)$ such that the series

$$\sum_1^{\infty} \rho_n(x)$$

converges. We shall show later how such an estimate can be derived from a more general integral inequality than (12).

Let us consider another illustration of the iteration procedure. The problem is to find a solution $x(t)$ of

$$\ddot{x} + x = f(x, \dot{x}, t)$$

such that $x(0) = A$ and $x'(0) = 0$. Here we may consider Lx as $\ddot{x} + x$ and $Mx = f(x, \dot{x}, t)$. The inverse operator to L is easily found and we obtain

$$x = A \cos t + \int_0^t f[x(\tau), \dot{x}(\tau), \tau] \sin(t - \tau) d\tau .$$

We set up the customary iteration as follows:

$$x_0 = A \cos t$$

$$x_{n+1} = A \cos t + \int_0^t f[x_n(\tau), \dot{x}_n(\tau), \tau] \sin(t - \tau) d\tau .$$

But now, we see that the estimate for $x_{n+1}(t) - x_n(t)$ will depend also on an estimate for $\dot{x}_n(\tau) - \dot{x}_{n-1}(\tau)$.

The simplest way to avoid estimating the derivatives is to reformulate the problem in terms of the following system of two first-order differential equations:

$$\dot{x} = v$$

$$\dot{v} = -x + f(x,v,t)$$

with the conditions $x(0) = A$, $v(0) = 0$. It is easy to obtain the following iteration procedure:

$$x_{n+1}(t) = A\cos t + \int_0^t f[x_n(\tau), v_n(\tau), \tau]\sin(t-\tau)d\tau$$

(13)

$$v_{n+1}(t) = -A\sin t + \int_0^t f[x_n(\tau), v_n(\tau), \tau]\cos(t-\tau)d\tau$$

with $x_0(t) = A$, $v_0(t) = 0$. We use as a norm

$$\rho_{n+1}(t) = \max\left\{|x_{n+1}(t) - x_n(t)|, |v_{n+1}(t) - v_n(t)|\right\} .$$

Again, a condition on the derivatives

$$\frac{\partial f(\xi,\eta,\tau)}{\partial \xi}, \quad \frac{\partial f(\xi,\eta,\tau)}{\partial \eta}$$

is needed. If $m(\tau)$ is a bound for the sum of the absolute values of these derivatives, we can obtain from (13) the inequality

(14) $$\rho_{n+1}(t) \leq \int_0^t m(\tau)\rho_n(\tau)d\tau .$$

If this inequality were an equality, and $\rho_0(t)$ equalled a constant b_0, we would find that

$$\rho_n(t) = \frac{M(t)^n}{n!}b_0$$

where

$$M(t) = \int_0^t m(\tau)d\tau .$$

This suggests putting in (14)

$$\rho_n(t) = \frac{M(t)^n}{n!}b_n(t).$$

After this substitution, (14) becomes

$$\frac{M(t)^{n+1}}{(n+1)!}b_{n+1}(t) \leq \int_0^t \frac{M(\tau)^n}{n!}b_n(\tau)d\tau$$

Put $B_n = \max b_n(t)$ for $0 \leq t \leq a$. Then this inequality implies that

$$\frac{M(t)^{n+1}}{(n+1)!}b_{n+1}(t) \leq B_n \int_0^t \frac{M(\tau)^n}{n!}d\tau = B_n \frac{M(t)^{n+1}}{(n+1)!}$$

or

$$b_{n+1}(t) \leq B_n$$

for $0 \leq t \leq a$; consequently, $B_{n+1} \leq B_n \leq B_{n-1} \leq \cdots \leq B_1$. This shows that the sequence of B_n is bounded and then

(15) $$\rho_n(t) \leq B_1 \frac{M(t)^n}{n!} .$$

Note that a similar result can be obtained for (12). We have $m(t) = C$, $M(t) = Ct$ and

$$\rho_n(t) \leq B_1 \frac{(Ct)^n}{n!} .$$

Using the estimate given in (15), we find that

$$\sum_1^\infty \rho_n(t) \leq B_1 e^{M(t)},$$

a convergent result, which shows that the iteration procedure defined in (13) converges to a limit.

The iteration method leads to an important result if the operator K in (4) is linear. In such a case

$$u_{n+1} = g + \varepsilon K u_n = g + \varepsilon K(g + \varepsilon K u_{n-1})$$

$$= g + \varepsilon Kg + \varepsilon^2 K^2 u_{n-1} = \cdots = g + \varepsilon Kg + \varepsilon^2 K^2 g + \cdots + \varepsilon^n K^n g.$$

We thus obtain the formal Neumann series:

$$(16) \qquad u = g + \sum_{1}^{\infty} (\varepsilon K)^n g.$$

The series will converge if the operator εK is "small" enough. A sufficient condition for convergence is that the eigenvalue of εK with smallest absolute value have magnitude less than one.

It should be noted that the infinite series (16) could also be obtained by assuming u has a series expansion in powers of ε such as

$$u = v_0 + \sum_{1}^{\infty} \varepsilon^n v_n ,$$

substituting this in (4) and equating the coefficients of corresponding powers of ε on each side of the resulting equation.

Fredholm Expansion

The Neumann series given in (16) converges in general only for sufficiently small values of ε. Fredholm showed how to obtain an expansion for the solution of (4) which is valid for all values of ε for which the solution to (4) exists. Of course, the operator K must be assumed linear. We shall obtain this expansion in a formal manner and ignore the important question of proving the convergence of the expansion.

The solution of (4) may be written as

(17) $$u = (I - \varepsilon K)^{-1} g$$

where I is the identity operator. If we consider the case where K is a finite-dimensional matrix, the solution (17) breaks down when the determinant of $I - \varepsilon K$ is zero. Put

$$D(\varepsilon) = \det(I - \varepsilon K).$$

We shall state later the meaning of the determinant of a non-matrix operator.

Rewrite (17) as follows

(18) $$u = \left[I + \frac{\varepsilon D(K,\varepsilon)}{D(\varepsilon)} \right] g$$

where

$$I + \frac{\varepsilon D(K,\varepsilon)}{D(\varepsilon)} = (I - \varepsilon K)^{-1}.$$

This implies

(19) $$D(K,\varepsilon) = \frac{K D(\varepsilon)}{I - \varepsilon K}.$$

If K is a finite-dimensional matrix, the formula (19) suggests that $D(K,\varepsilon)$ would be regular for all values of ε because the singularities of $(I - \varepsilon K)^{-1}$ occur at the zeros of $D(\varepsilon)$ and would be cancelled by the factor $D(\varepsilon)$ in the numerator. It seems reasonable to expect, therefore, that both $D(K,\varepsilon)$ and $D(\varepsilon)$ would have expansions in powers of ε which would converge for all values of ε.

Before we obtain these expansions, we wish to define the determinant of a general operator. If ε is small enough, we may define $\log(I - \varepsilon K)$ by the power series

$$-\sum_{1}^{\infty} \frac{(\varepsilon K)^n}{n}.$$

If $I - \varepsilon K$ is a matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$, then

$$\det(I - \varepsilon K) = \lambda_1, \lambda_2, \cdots, \lambda_m = \exp\left(\sum_1^m \log\lambda_y\right).$$

But $\log(I - \varepsilon K)$ has the eigenvalues $\log\lambda_1, \cdots, \log\lambda_m$ and

$$\operatorname{trace} \log(I - \varepsilon K) = \sum_1^m \log\lambda_y;$$

therefore, for a matrix we obtain the formula

$$(20) \qquad \det(I - \varepsilon K) = \exp[\operatorname{tr} \log(I - \varepsilon K)],$$

where tr stands for trace.

Formula (20) can be used to define the determinant of $(I - \varepsilon K)$ for general operators K if the trace of these operators can be defined. We shall define trace only for integral operators. Suppose

$$(21) \qquad Kg = \int_a^b k(x,y)g(y)dy$$

then

$$(22) \qquad \operatorname{tr} K = \int_a^b k(y,y)dy.$$

Suppose

$$(23) \qquad D(\varepsilon) = \sum_0^\infty \frac{(-)^n}{n!} c_n \varepsilon^n$$

and

$$(24) \qquad D(K,\varepsilon) = \sum_0^\infty \frac{(-)^n}{n!} B_n \varepsilon^n$$

where c_n are constants and B_n are operators. Using (20), we find that

$$(25) \qquad \log D(\varepsilon) = \operatorname{tr}[\log(I - \varepsilon K)] = -\operatorname{tr} \sum_1^\infty \frac{(\varepsilon K)^n}{n} = -\sum_1^\infty \frac{\varepsilon^n}{n} k_n$$

where

(26) $$k_n = \operatorname{tr} K^n.$$

Differentiating (25), we get

(27) $$\frac{D'(\varepsilon)}{D(\varepsilon)} = -\sum_1^\infty \varepsilon^{n-1} k_n = -\operatorname{tr} \frac{K}{I - \varepsilon K},$$

or

$$D'(\varepsilon) = \sum_0^\infty \frac{(-)^n}{(n-1)!} c_n \varepsilon^{n-1} = -\left(\sum_1^\infty \varepsilon^{n-1} k_n\right) D(\varepsilon).$$

Multiplying the two series on the right-hand side and equating coefficients of corresponding powers of ε, we get a set of linear equations from which c_n could be determined. Soon, we shall obtain a more useful formula for c_n.

From (19) we have

$$D(K,\varepsilon) - \varepsilon K D(K,\varepsilon) = K D(\varepsilon).$$

Substituting the series (23) and (24) in this equation and comparing the coefficients of corresponding powers of ε, we get

(28) $$B_n + nKB_{n-1} = c_n K$$

for $n = 0,1,2,\ldots.$ Note that from (23), $c_0 = 1$, $B_0 = K$. Again use (19) and take the trace of both sides. We find

$$\operatorname{tr} D(K,\varepsilon) = D(\varepsilon) \operatorname{tr} \frac{K}{I - \varepsilon K} = -D'(\varepsilon)$$

by the use of (27). Comparing corresponding powers of ε, we obtain the promised formula for c_n, namely,

(29) $$c_n = \operatorname{tr} B_{n-1}.$$

Formulas (28) and (29) enable us to determine c_n and B_n by succession. We find

$$c_0 = 1,\ B_0 = K$$

(30) $$c_1 = k_1,\ B_1 = k_1K - K^2$$

$$c_2 = k_1^2 - k_2,\ B_2 = (k_1^2 - k_2)K - 2KB_1,$$

etc. This is as far as we can go in the general case. However, if we assume that K is an integral operator, as Fredholm did, we can give some elegant determinantal representations for B_n and c_n.

We assume

$$Kg = \int_a^b K(x,y)g(y)dy$$

where $K(x,y)$ is called the kernel of the integral operator K. If $B_n(x,y)$ is the kernel for B_n, then (28) becomes

(31) $$B_n(x,y) + n\int_a^b K(x,z)B_{n-1}(z,y)dz = c_nK(x,y),$$

and (29) becomes

(32) $$c_n = \int_a^b B_{n-1}(y,y)dy \ .$$

With these notations, the formulas (30) may be written as follows:

$$B_1(t_0,t_2) = \int_a^b dt_1 \begin{vmatrix} K(t_0,t_2) & K(t_0,t_1) \\ K(t_1,t_2) & K(t_1,t_1) \end{vmatrix},\quad c_1 = \int_a^b K(t_1,t_1)dt,$$

$$B_2(t_0,t_3) = \int_a^b \int_a^b dt_1 dt_2 \begin{vmatrix} K(t_0,t_3) & K(t_0,t_1) & K(t_0,t_2) \\ K(t_1,t_3) & K(t_1,t_1) & K(t_1,t_2) \\ K(t_2,t_3) & K(t_2,t_1) & K(t_2,t_2) \end{vmatrix}, \quad c_2 = \int_a^b t_2 B_1(t_2,t_2).$$

It is clear what the general form of $B_n(t_0,t_{n+1})$ and c_n should be. By mathematical induction using (31) and (32), it is not difficult to prove the formulas correct. We leave the details to the reader.

Perturbation of Eigenvalues

Consider the problem of finding the eigenvalues of the linear operator $L + \varepsilon M$ if the eigenvalues and eigenfunctions of L are known. The problem is to find values of λ for which there exist non-zero solutions of

(33) $\qquad (L + \varepsilon M)u = \lambda u.$

Here both L and M are linear operators and we assume L is self-adjoint.

Suppose u_0 is an eigenfunction of L corresponding to the eigenvalue λ_0 and we wish to find the eigenvalue of $L + \varepsilon M$ which approaches λ_0 as ε approaches zero. Assume

$$\lambda = \lambda_0 + \sum_1^\infty \varepsilon^k \lambda_k$$

and

$$u = u_0 + \sum_1^\infty \varepsilon^k u_k$$

where the λ_k and $u_k (k \geq 1)$ are to be determined. Substitute these expansions for λ and u into (33) and equate the coefficients of the corresponding powers of ε on each side of the resulting equation. In this way we obtain the following set of equations:

(34) $$Lu_0 = \lambda_0 u_0$$

(35) $$(L - \lambda_0)u_1 = (\lambda_1 - M)u_0$$

(36) $$(L - \lambda_0)u_2 = (\lambda_1 - M)u_1 + \lambda_2 u_0,$$

etc.

The first of these equations is automatically satisfied by our choice of λ_0 and u_0. In the second equation we do not know λ_1 and u_1. In general, this equation has no solution because the homogeneous equation

$$(L - \lambda_0)v = 0$$

has a non-zero solution $v = u_0$. A necessary condition for the existence of a solution of the second equation is that the right-hand side be orthogonal to u_0, that is,

$$\int u_0(\lambda_1 - M)u_0 = 0.$$

This equation implies

$$\lambda_1 \int u_0^2 = \int u_0 M u_0$$

and thus defines λ_1. With this definition of λ_1, equation (35) will in general have an infinite number of solutions because to any particular solution v_1 of (35) we may add $\alpha_1 u_0$ where α_1 is an arbitrary constant. We choose α_1 so that if

$$u_1 = v_1 + \alpha_1 u_0$$

then

$$\int uu_1 = 0.$$

This will be satisfied if

$$\alpha_1 \int u_0^2 = - \int u_0 v_1 .$$

Equation (36) may be treated similarly. It will have a solution only if the right-hand side is orthogonal to u_0, that is, if

$$\int u_0(\lambda_1 - M)u_1 + \lambda_2 \int u_0^2 = 0.$$

This equation determines λ_2. If v_2 is now a particular solution of (36), then

$$u_2 = v_2 + \alpha_2 u_0$$

is also a solution. We choose α_2 so that

$$\int u_0 u_2 = 0,$$

that is, α_2 satisfies the equation

$$\int v_2 u_0 + \alpha_2 \int u_0^2 = 0.$$

In a similar way, we can successively determine λ_n, u_n and thus obtain the complete expansion for λ and u. Under fairly general conditions on the operators L and M, it can be shown that the expansions converge for sufficiently small values of ε. Notice that, because of the way the α's were chosen, the eigenfunction u determined by the expansion is such that

$$\int uu_0 = \int u_0^2 .$$

Chapter II

SINGULAR PERTURBATIONS

Introduction

In the last section of Chapter I we have discussed a method for finding the eigenvalues of a complicated operator as perturbations of the eigenvalues for a simpler operator. In the present chapter we shall discuss operators for which the proposed method breaks down and for which a more sophisticated approach is needed. The type of difficulty we shall treat is illustrated in the following example given by Lord Rayleigh:

Find the values of λ for which there exist non-zero solutions of

$$\varepsilon u^{IV} - u'' = \lambda u \tag{1}$$

such that $u(0) = u'(0) = u(1) = u'(1) = 0$.

This eigenvalue problem when treated by the method of the preceding chapter leads to the equation

$$- u'' = \lambda u \tag{2}$$

with the boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$. For any value of λ, the only solution of (2) satisfying all four boundary conditions is u identically zero and the perturbation method breaks down before we can even begin.

A problem such as (1) for which the simple perturbation techniques explained in Chapter I do not work will be called a singular perturbation problem. The singularity is obviously due to the fact that the zeroth order equation, that is, the equation for $\varepsilon = 0$, is of lower order than the original equation (1) and we are troubled by a plethora of boundary conditions. This chapter will treat the general case of an operator $L + \varepsilon M$, where L is a differential operator of the nth order, M a differential operator of the mth order and $m > n$. We shall discuss both the eigenvalue problem for $L + \varepsilon M$ and the solution of the non-homogeneous equation

$$(L + \varepsilon M)u = f,$$

and also give a priori rules for determining which boundary conditions will be used.

The Exact Solution for the Rayleigh Problem

Because equation (1) has constant coefficients, it can be solved exactly. Assume $u = e^{rx}$ is a solution of (1); then r must satisfy the equation

$$\varepsilon r^4 - r^2 = \lambda.$$

Solving for r^2, we get

$$r^2 = \frac{1 \pm \sqrt{1 + 4\lambda\varepsilon}}{2\varepsilon}. \tag{3}$$

We assume ε small and positive and equal to η^2 and consider first the positive square root in (3). We find

$$r^2 = \frac{2 + 2\lambda\eta^2 + O(\eta^4)}{2\eta^2} = \eta^{-2}[1 + \lambda\eta^2 + O(\eta^4)].$$

Taking the square root of this, we get

$$(4) \qquad r = \pm r_1 = \pm \eta^{-1}[1 + \tfrac{1}{2}\lambda\eta^2 + O(\eta^4)].$$

From the negative square root in (3) we find

$$r^2 = \frac{1 - [1 + 2\lambda\eta^2 - 2\lambda^2\eta^4 + O(\eta^6)]}{2\eta^2}$$

$$= -\lambda(1 - \lambda\eta^2) + O(\eta^4)$$

and

$$(5) \qquad r = \pm i r_2 = \pm i\lambda^{1/2}(1 - \tfrac{\lambda}{2}\eta^2) + O(\eta^4).$$

Using (4) and (5), the solutions of (1) are linear combinations of

$$e^{r_1 x}, \quad e^{-r_1 x}, \quad \sin r_2 x, \quad \cos r_2 x.$$

Let us take

$$u = \alpha \sin r_2 x + \beta \cos r_2 x + \gamma_+ e^{r_1 x} + \gamma_- e^{-r_1 x}$$

where α, β, γ_+, and γ_- are constants that will be determined to fit the boundary conditions. We have

$$u(0) = 0 = 0 + \beta + \gamma_+ + \gamma_- ,$$

$$u'(0) = 0 = r_2\alpha + 0 + r_1\gamma_+ - r_1\gamma_- ,$$

$$u(1) = 0 = \alpha \sin r_2 + \beta \cos r_2 + \gamma_+ e^{r_1} + \gamma_- e^{-r_1},$$

$$u'(1) = 0 = r_2\alpha \cos r_2 - r_2 \sin r_2 + r_1\gamma_+ e^{r_1} - r_1\gamma_- e^{-r_1}.$$

These four linear equations for α, β, γ_+ and γ_- have a non-zero solution if and only if the determinant of the coefficients is zero, that is, if and only if

$$(6)\qquad \begin{vmatrix} 0 & 1 & 1 & 1 \\ r_2 & 0 & r_1 & -r_1 \\ \sin r_2 & \cos r_2 & e^{r_1} & e^{-r_1} \\ r_2 \cos r_2 & -r_2 \sin r_2 & r_1 e^{r_1} & -r_1 e^{-r_1} \end{vmatrix} = 0 .$$

Note that in (4) r_1 has been defined so that its real part is positive for small values of η. This implies that e^{-r_1} goes exponentially to zero as η goes to zero. This fact eliminates the last two elements in the fourth column of (6). Multiply the third column of (6) by e^{-r_1} so as to get exponentially small factors in the first two elements of the third column. Because of this condition (6) is equivalent to the following:

$$0 = r_1^2 \begin{vmatrix} 0 & 1 & 0 & 1 \\ r_2 & 0 & 0 & -r_1 \\ \sin r_2 & \cos r_2 & 1 & 0 \\ r_2 \cos r_2 & -r_2 \sin r_2 & r_1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ \sin r_2 & 0 \end{vmatrix} + O(r_1^{-1}) .$$

Since r_1 approaches infinity as η approaches zero, we conclude that in the limit we must have

$$\sin r_2 = 0.$$

However, from (5), $r_2 = \lambda^{1/2}$ in the limit; therefore,

$$\sin \sqrt{\lambda} = 0, \tag{7}$$

or $\lambda = n^2\pi^2$ for $n \geq 1$.

This result shows that in the limit the eigenvalue problem for (1) goes into the eigenvalue problem for (2) with the boundary conditions $u(0) = u(1) = 0$. Using (7) and the boundary conditions, we can see that also the eigenfunction of (1) approaches the eigenfunction of (2) with zero boundary conditions, namely, $\sin n\pi x$.

Boundary Layer Effect

We have just shown that as ε approaches zero the eigenfunctions for (1) approach $\sin n\pi x$. This seems paradoxical because, even though $u = \sin n\pi x$ satisfies the limiting equation (2) and some of the boundary conditions, it does not satisfy the conditions $u'(0) = u'(1) = 0$. We proceed to investigate how the boundary condition $u'(0) = 0$ is lost as ε approaches zero.

To study the behavior of the solutions of (1) near $x = 0$, we "stretch" the x-axis by putting

$$x = \eta t$$

in (1). The equation becomes

$$\dddot{u} - \ddot{u} = \lambda\eta^2 u \tag{8}$$

where dots denote differentiation with respect to t. The boundary conditions at $t = 0$ are still $u(0) = u'(0) = 0$ but the boundary conditions at $x = 1$ become conditions at $t = \eta^{-1}$ which approaches infinity. Notice, however, that $t = \infty$ does not correspond to one value of x but to all values of x such that $\eta^{-1}x$ is unbounded. Because of this, we cannot apply the conditions at $x = 1$. Instead, we try to find a solution of (8) or, more precisely, of the limiting form of (8), that is,

$$\ddddot{u} - \ddot{u} = 0, \tag{9}$$

such that u will, for large values of t, fit in or match the function $u_0 = \sin \pi nx = \sin \pi n\eta t$.

Any solution of (9) that satisfies the boundary conditions $u(0) = u'(0) = 0$ can be written as

$$u = \alpha(1 + t - e^{t}) + \beta(1 - t - e^{-t}) \tag{10}$$

where α and β are arbitrary constants. To have this solution fit into the function $\sin \pi n\eta t$, it is clear that, first, α must be zero. Next, if we let t go to infinity but in such a fashion that x is small (this can be done by taking $t = \eta^{-1/2}$ and $x = \eta^{1/2}$), the function $\sin \pi n\eta t$ can be approximated by $\pi n\eta t$; consequently, for u to behave like $\pi n\eta t$ when t is large, we must take $\beta = -\pi n\eta$. We therefore conclude that the appropriate solution of (9) is

$$u = \pi n\eta(e^{-t} - 1 + t) = \pi nx + \pi n\eta(e^{-x/\eta} - 1). \tag{11}$$

A similar argument can be made for the neighborhood of $x = 1$. If we introduce the stretched coordinate

$$1 - x = \eta\tau$$

into (1) and go to the limit $\eta = 0$, we again obtain the equation

$$u^{IV} - u'' = 0 \tag{12}$$

with the conditions $u = u' = 0$ for $\tau = 0$ and u behaves like $\sin \pi n(1-\eta\tau)$ as τ approaches infinity. The solution of (12) satisfying these conditions is

$$\begin{aligned} u &= \cos n\pi\,(1 - \tau - e^{-\tau})\pi n\eta \\ &= \cos n\pi[\pi n\eta(1 - e^{(1-x)/\eta}) - \pi n(1 - x)]. \end{aligned} \tag{13}$$

A study of (11) and (13) explains the paradoxical behavior of the zeroth-order eigenfunctions at the boundary conditions. The actual eigenfunction, for $\eta \neq 0$, behaves like (11) in the neighborhood of $x = 0$ and like (13) in the neighborhood of $x = 1$; thus, the actual eigenfunction satisfies all four boundary conditions. But as both (11) and (13) indicate, as x moves away from either endpoint the exponential terms damp out very quickly and the actual eigenfunction goes over into $\sin n\pi x$, the zeroth-order eigenfunction. This situation is illustrated, on an exaggerated scale, in Fig. 1.

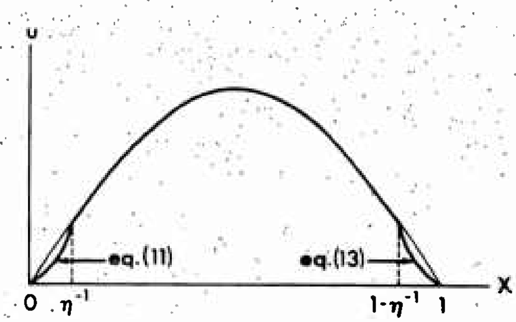

FIGURE 1

The behavior of the curve at $x = 0$ and $x = 1$ is typical of that which occurs in boundary layer problems in fluid flow. We shall call boundary layer effect any such occurrence in which a very sudden transition occurs between two parts of a solution.

First Order Correction to the Eigenvalues

To obtain an improved value for the eigenvalues of (1), we use (8) after replacing u on the right-hand side by the zeroth order approximation $\sin\sqrt{\lambda}\, x$ thus obtaining

$$\ddddot{u} - \ddot{u} = \eta^2\lambda \sin\sqrt{\lambda}\,\eta t\,. \tag{14}$$

We solve this equation in the neighborhood of $x = 0$ with the boundary conditions $u = u' = 0$ at $t = 0$ and u matches $\sin n\pi\eta t$ as t approaches infinity. A particular solution of this equation is the function

$$\frac{\sin\sqrt{\lambda}\,\eta t}{1 + \eta^2\lambda}\,.$$

For the complementary solution we shall use the functions e^t, e^{-t} and the functions $\cos n\pi\eta t$ and $\sin n\pi\eta t$ instead of the functions 1 and t. We choose the cosine and sine functions because they are the exact solutions of the limit equation (2). We therefore put

$$u = \frac{\sin\sqrt{\lambda}\,\eta t}{1 + \eta^2\lambda} + \alpha\cos n\pi\eta t + \beta\sin n\pi\eta t + \gamma_+ e^t + \gamma_- e^{-t}$$

where α, β, γ_+, γ_- are to be determined to fit the boundary conditions. In order that u not go to infinity as t goes to infinity, we must have $\gamma_+ = 0$. To satisfy the conditions $u = 0$ and $u' = 0$, we find

$$\alpha + \gamma_- = 0,$$

$$\sqrt{\lambda}\,\eta\left(\beta + \frac{1}{1 + \eta^2\lambda}\right) - \gamma_- = 0.$$

To have u match with $\sin n\pi\eta t$ as t goes to infinity, we must have

$$\beta + (1 + \eta^2\lambda)^{-1} = 1,$$

and $\sqrt{\lambda}$ approach $n\pi$. Neglecting terms of order η^2, we conclude that

$$(15) \qquad \begin{aligned} u &= \frac{\sin\sqrt{\lambda}\,\eta t}{1 + \eta^2\lambda} - \sqrt{\lambda}\,\eta(\cos n\pi\eta t - e^{-t}) \\ &= \frac{\sin\sqrt{\lambda}\,x}{1 + \eta^2\lambda} - \sqrt{\lambda}\,\eta(\cos n\pi x - e^{-x/\eta}). \end{aligned}$$

Formula (15) gives the correct (to order η^2) behavior of the eigenfunction near $x = 0$ and also in the middle of the interval $0 \leq x \leq 1$. To find the correct behavior of the eigenfunction near $x = 1$, we put $1 - x = \eta\tau$ in (1) and obtain

$$\dddot{u} - \ddot{u} = \eta^2\lambda u.$$

We desire the solution of this which satisfies the conditions $u = u' = 0$ at $\tau = 0$ and again matches $\sin\sqrt{\lambda}\,x = \sin\sqrt{\lambda}\,(1 - \eta\tau)$ as τ approaches infinity. Just as in (15), we find that

$$(16) \qquad \begin{aligned} u &= \frac{\sin\sqrt{\lambda}\,\eta\tau}{\cos\sqrt{\lambda}\,(1+\eta^2\lambda)} + \eta\sqrt{\lambda}\,(\cos n\pi\eta\tau - e^{-\tau}) \\ &= \frac{\sin\sqrt{\lambda}\,(1-x)}{(1+\eta^2\lambda)\cos\sqrt{\lambda}} + \eta\sqrt{\lambda}\,[\cos n\pi(1-x) - e^{-(1-x)/\eta}] \end{aligned}$$

correct to terms of order η^2.

In order that (15) and (16) represent the same solution we must have

$$\begin{aligned} &\sin\sqrt{\lambda}\,x - \eta\sqrt{\lambda}\cos n\pi x \\ &= \frac{\sin\sqrt{\lambda}\,(1-x)}{\cos\sqrt{\lambda}} + \eta\sqrt{\lambda}\cos n\pi\cos n\pi x, \end{aligned}$$

or

$$(17) \qquad \cos\sqrt{\lambda}\,x\,\sin\sqrt{\lambda} = \eta\sqrt{\lambda}\cos n\pi x(1 + \cos\sqrt{\lambda}\cos n\pi).$$

Clearly, $\sqrt{\lambda} = n\pi$ in the lowest order. If we put

$$\sqrt{\lambda} = n\pi + \lambda_1 \eta$$

in (17), we find

$$\sin \lambda_1 \eta = 2\eta n\pi$$

correct to order η^2; therefore, $\lambda_1 = 2n\pi$ and

$$\sqrt{\lambda} = n\pi(1 + 2\eta).$$

The eigenfunction to this order is

$$\sin n\pi(1 + 2\eta)x.$$

The General Case

Let us consider over the interval $a \leq x \leq b$ the general differential equation

$$(L + \varepsilon M)u = f \tag{18}$$

where

$$Lu = \sum_0^n p_k(x)u^{(k)},$$

$$Mu = \sum_0^m q_k(x)u^{(k)},$$

and $m > n$. If $f = 0$ and $L = L_0 - \lambda$ where L_0 is a differential operator of the same form as L, then (18) becomes an eigenvalue problem; otherwise, we shall call it the non-homogeneous problem. In both cases, to make the problem precise we must have m boundary conditions which we represent as follows:

$$
(19) \quad
\begin{aligned}
B_1(u) &= u^{(\alpha_1)}(a) + \sum_{j=1}^{\alpha_1 - 1} \alpha_{1j} u^{(j)}(a) = 0 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_s < m \\
B_s(u) &= u^{(\alpha_s)}(a) + \sum_{j=1}^{\alpha_s - 1} \alpha_{sj} u^{(j)}(a) = 0 \\
B_{s+1}(u) &= u^{(\beta_1)}(b) + \sum_{j=1}^{\beta_1 - 1} \beta_{1j} u^{(j)}(b) = 0 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad 0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_t < m \\
B_m(u) &= u^{(\beta_t)}(b) + \sum_{j=1}^{\beta_t - 1} \beta_{tj} u^{(j)}(b) = 0 .
\end{aligned}
$$

Here α_i, α_{ij}, β_i, β_{ij} are given constants and $m-n = t$.

When ε goes to zero in (18), the differential equation reduces to a differential equation of lower order, namely,

$$Lu = f$$

and, just as in the Rayleigh example, we must determine which of the boundary conditions in (19) will be retained. In order to determine this, we need to know the asymptotic behavior of the solutions of the homogeneous equation

$$(20) \qquad Lu + \varepsilon Mu = 0.$$

It is convenient to put $\varepsilon = \eta^t$. We write (20) as

$$(21) \qquad Lu = -\eta^t Mu$$

and we shall show that this equation has n solutions which are analytic power series in η. (It is assumed that $p_u(x)$ and $q_u(x)$ are analytic functions of x in $a \le x \le b$ and $p_n(x) \ne 0$ in that interval.) We indicate the idea. Let $u_{j0}(x)$ be any solution of

$$Lu = 0$$

neglecting boundary conditions. Put

$$u_j(x) = u_{j0}(x) + \sum_{k=1}^{\infty} \eta^k u_{jk}(x), \qquad 1 \le j \le n \tag{22}$$

in (21) and compare corresponding powers of η. We find

$$Lu_{jk}(x) = -Mu_{j,k-t}(x).$$

Let $u_{jk}(x)$ be the solution of this equation such that the solution and its first $n-1$ derivatives are zero at $x = a$. It is clear that this solution exists and is unique. Thus, starting with $u_{j0}(x)$, we can successively find all the $u_{jk}(x)$ and thus obtain the expansion (22) for $u_j(x)$. It can be shown that the expansion is an asymptotic one. By choosing $u_{j0}(x)$, $(1 \le j \le n)$ as n linearly independent solutions of $Lu = 0$, it is clear that we shall get n linearly independent solutions of (21).

To obtain t other linearly independent solutions of (21), we begin by considering the characteristic equation

$$p_n(x) + r^t q_n(x) = 0. \tag{23}$$

If we assume $q_n(x) \ne 0$ for $a \le x \le b$, this equation has for every x in (a,b) t solutions, namely, the t t^{th} roots of $-p_n(x)/q_n(x)$. We shall assume that

$$\text{Re } r(x) \ne 0$$

for every x in (a,b) and any root of (23).

For any root $r(x)$ of (23) we shall construct an asymptotic expansion of a solution to (21). For simplicity of exposition, we shall show how the first two terms of the expansion can be obtained. The infinite expansion can be obtained in a similar fashion. For further details we refer to a paper by Jurgen Moser in Communications on Pure and Applied Mathematics, vol. VIII, 1955, pp. 251-279.

It is important to distinguish between $\operatorname{Re} r(x) > 0$ and $\operatorname{Re} r(x) < 0$. We shall denote a solution of (21) corresponding to the case $\operatorname{Re} r(x) > 0$ by $v(x)$ and a solution corresponding to the case $\operatorname{Re} r(x) < 0$ by $w(x)$. Put

$$h(x) = \int_c^x r(\xi)\,d\xi$$

where $a \leq c \leq b$. Let

$$v(x) = \eta \exp[\eta^{-1}h(x)] \cdot [v_0(x) + \eta v_1(x) + \cdots .$$

Note that

$$v^{(j)}(x) = \eta^{n-j}\exp[\eta^{-1}h(x)]\cdot\{r^j(x)v_0(x) + \eta[r^j v_1 + \frac{j(j-1)}{2} r^{j-2}r'v_0 + jr^{j-1}v_1'] + \cdots\}.$$

Substituting this in (21) and dividing by $\exp[\eta^{-1}h(x)]$, we get

$$p_n(x)\{r^n v_0 + \eta[r^n v_1 + \frac{n(n-1)}{2} r^{n-2}r'v_0 + nr^{n-1}v_0']\} + \eta p_{n-1}(x)r^{n-1}v_0 +$$

$$+ q_m(x)\{r^m v_0 + \eta[r^m v_1 + \frac{m(m-1)}{2} r^{m-2}r'v_0 + mr^{m-1}v_0']\} + \eta p_{m-1}(x)r^{m-1}v_0 = 0$$

after neglecting terms of order η^2. This implies that

$$[p_n(x)r^n + q_m(x)r^m]v_0 = 0,$$

$$v_1(p_n r^n + q_m r^m) + \frac{v_0 r'}{2}[n(n-1)p_n r^{n-2} + m(m-1)q_m r^{m-2}] + [p_{n-1}r^{n-1} +$$

$$+ q_{m-1}r^{m-1}]v_0 + v_0'(np_n r^{n-1} + mq_m r^{m-1}) = 0.$$

Because of equation (23), the first of these equations is satisfied for arbitrary $v_0(x)$ and the second equation reduces to

$$(24) \qquad v_0'[np_n(x)r^{n-1} + mq_m(x)r^{m-1}] + v_0[\frac{n(n-1)}{2} p_n r^{n-2} r' +$$

$$+ \frac{m(m-1)}{2} q_m r^{m-2} r' + p_{n-1} r^{n-1} + q_{m-1} r^{m-1}] = 0.$$

The coefficient of v_0' in (24) can never be zero because, if it were, r would be a multiple root of (23) -- which is known to be impossible; consequently, (24) is a linear ordinary differential equation for $v_0(x)$ which can be solved by separation of variables.

It is easy to show that a similar procedure will enable us to determine a complete expansion for $v(x)$. The proof that it is an asymptotic expansion of a solution of (21) will be found in the paper of J. Moser cited previously.

The Lost Boundary Conditions

Let us return to (18). For the non-homogeneous problem let u_0 be a particular solution of (18). Then a solution $u(x)$ of (21) must be found such that $u+u_0$ satisfies the boundary conditions (19). Similarly, in the eigenvalue problem a solution u of (21) must be found satisfying the boundary conditions (19). We wish to determine which boundary conditions will be retained in the limit ε or η equals zero.

The previous sections have shown that the solutions of (21) can be divided among three classes: the functions $u_1(x),\ldots,u_n(x)$ which were obtained in (22) by starting with solutions of $Lu = 0$, the functions $v_1(x),\ldots,v_p(x)$, where p is an integer not greater than t, which were obtained in the preceding section on the assumption $\operatorname{Re} r(x) > 0$, and, finally, the functions $w_1(x),\ldots,w_{t-p}(x)$, which are obtained similarly to the $v(x)$ but on the assumption that $\operatorname{Re} r(x) < 0$.

If u is a solution of (21), we may write

$$u = \alpha_1 u_1 + \cdots + \alpha_n u_n + \beta_1 v_1 + \cdots + \beta_p v_p + \gamma_1 w_1 + \cdots + \gamma_{t-p} w_{t-p}$$

and then the boundary conditions will give us equations such as these:

$$\begin{aligned} &\alpha_1 B_k(u_1) + \cdots + \alpha_n B_k(u_n) + \beta_1 B_k(v_1) + \cdots + \beta_p B_k(v_p) + \\ (25) \qquad &+ \gamma_1 B_k(w_1) + \cdots + \gamma_{t-p} B_k(w_{t-p}) = B_k(u_0), \qquad 1 \leq k \leq m. \end{aligned}$$

The existence of a solution of these linear equations will depend on the behavior of the determinant of the coefficients, that is, on the determinant

$$\Delta = \begin{vmatrix} B_1(u_1) & \cdots & B_1(u_n) & B_1(v_1) & \cdots & B_1(v_p) & B_1(w_1) & \cdots & B_1(w_{t-p}) \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ B_s(u_1) & \cdots & B_s(u_n) & B_s(v_1) & \cdots & B_s(v_p) & B_s(w_1) & \cdots & B_s(w_{t-p}) \\ B_{s+1}(u_1) & \cdots & B_{s+1}(u_n) & B_{s+1}(v_1) & \cdots & B_{s+1}(v_p) & B_{s+1}(w_1) & \cdots & B_{s+1}(w_{t-p}) \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ B_m(u_1) & \cdots & B_m(u_n) & B_m(v_1) & \cdots & B_m(v_p) & B_m(w_1) & \cdots & B_m(w_{t-p}) \end{vmatrix}$$

Suppose that $v_k(x)$ has the exponential factor $\exp[\eta^{-1}h_k(x)]$ where

$$h_k(x) = \int^x r_k(\xi)\,d\xi$$

and $\operatorname{Re} r_k(x) > 0$; then the quantity $\exp[\eta^{-1}h_k(b)]$ will be exponentially large compared to $\exp[\eta^{-1}h_k(a)]$. If we multiply the column of Δ containing v_k by $\exp[-\eta^{-1}h_k(b)]$, the first s rows will have the factor

$$\exp[\eta^{-1}h_k(a)-\eta^{-1}h_k(b)] = \exp[-\eta^{-1}\int_a^b r(\xi)\,d\xi]$$

which goes exponentially to zero as η goes to zero. This implies that the first s rows of the v-columns will go to zero.

Similarly, by multiplying the w-columns by $\exp[-\eta^{-1}h(a)]$, the last $m-s$ rows will have the factor

$$\exp[\eta^{-1}h(b)-\eta^{-1}h(a)] = \exp[-\eta^{-1}\int_a^b r(\xi)\,d\xi]$$

which goes exponentially to zero as η goes to zero. This implies that the last $m-s$ rows of the w-columns will go to zero.

There is one further fact to be noticed. The jth derivative of $v(x)$ or of $w(x)$ has a factor η^{n-j} because of the differentiation of the exponential. As η goes to zero, this implies that $B_{s-k}(v)$ goes to zero compared to $B_s(v)$ for $1 \leq k \leq s-1$ and $B_{m-k}(w)$ goes to zero compared to $B_m(w)$ for $1 \leq k \leq m-s-1$. Using these facts we see that the leading term in Δ is obtained by expanding the following determinant:

$$
(26)\qquad \begin{vmatrix}
B_1(u_1) & \cdots & B_1(u_n) & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
B_s(u_1) & \cdots & B_s(u_n) & 0 & \cdots & 0 & B_s'(w_1) & \cdots & B_s'(w_{t-p}) \\
B_{s+1}(u_1) & \cdots & B_{s+1}(u_n) & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
B_m(u_1) & \cdots & B_m(u_n) & B_m'(v_1) & \cdots & B_m'(v_p) & 0 & \cdots & 0
\end{vmatrix} .
$$

Here the primed B_s and B_m are the B_s and B_m modified by the elimination of the exponential factor and the power of η. Clearly, the expansion of this determinant will not contain any term involving $B_s(u_k)$ or $B_m(u_k)$. This means the boundary conditions of highest derivative order at a and b must be discarded and only $m-2$ conditions are retained.

The determinant in (26) may be identically zero because there are too many zero rows. This will always happen if $n < m-2$. In that case in the expansion of Δ we must retain row m and row $m-1$ for the v's and row s and row $s-1$ for the w's. This would imply discarding the two highest derivative boundary conditions for both $x = a$ and $x = b$. This procedure continues until we obtain a non-identically zero determinant or, what will be equivalent, exactly n boundary conditions retained.

An examination of this method leads to the following rule:

<u>Discard a boundary condition at an endpoint for each solution of (21) that becomes exponentially large there (relative to its behavior at the other endpoint). Discard the boundary conditions in decreasing order of magnitude of the highest derivative present in the boundary condition.</u>

For the present problem since there are p functions $v(x)$ which become exponentially large at $x = b$, we discard boundary conditions B_m, B_{m-1}, ..., B_{m-p+1}, and since there are $t - p$ functions $w(x)$ which become exponentially large at $x = a$, we discard boundary conditions B_s, B_{s-1}, ..., $B_{s-t+p-1}$. The count is right because t conditions will be discarded and $m - t = n$ will be retained.

What happens if there are more exponentially large functions at an endpoint than there are boundary conditions? For example, what happens if $s - t + p < 1$? In that case, the solution of (18) will not approach a limit as ε goes to zero. Consider the following example:

(27) $$\varepsilon u'' + au' = 1$$

with $u(0) = u'(0) = 0$. The solution of this equation is

(28) $$u = \frac{x}{a} + \frac{\varepsilon}{a^2}(-1 + e^{-ax/\varepsilon}).$$

If $a > 0$, so that the solution $e^{-ax/\varepsilon}$ is exponentially large at $x = 0$ (compared to its value $e^{-a/\varepsilon}$ at $x = 1$), then (28) approaches the limit x/a. This limit is also the solution of the limit of equation (27), namely,

$$au' = 1$$

with the highest derivative boundary condition $(u' = 0)$ discarded and the other boundary condition $(u = 0)$ retained.

If $a < 0$, the solution $e^{-ax/\varepsilon}$ is exponentially large at $x = 1$; but there is no boundary condition at $x = 1$. Note, however, that (28) does not converge to a limit in this case.

Chapter III

RELAXATION OSCILLATIONS

An interesting type of singular perturbation occurs in the study of the Van der Pol equation

$$y'' - \mu(1 - y^2)y' + y = 0 \tag{1}$$

when the parameter μ is very large. If we change the independent variable from t to μt and put $\varepsilon = \mu^{-2}$, equation (1) becomes

$$\varepsilon\ddot{y} - (1 - y^2)\dot{y} + y = 0 \tag{2}$$

where dots denote differentiation with respect to the new variable. Put $y = \dot{x}$ in (2) and integrate to get the following form of the equation:

$$\varepsilon\ddot{x} - (\dot{x} - \frac{\dot{x}^3}{3}) + x = 0. \tag{3}$$

We shall show that (3) has periodic solutions and we shall find the period T asymptotically in ε for small values of ε. From the form of (3) it is clear that for $\varepsilon = 0$ the differential equation reduces to a first order equation which has no periodic solution; consequently, the perturbation around $\varepsilon = 0$ is singular.

The analysis of (3) becomes simpler when time is eliminated. Since $y = \dot{x}$, then $\ddot{x} = y\,dy/dx$ and (3) can be written as

$$\varepsilon\frac{dy}{dx} = \frac{F(y) - x}{y} \tag{4}$$

where

$$F(y) = y - y^3/3. \tag{5}$$

Let us show that any trajectory of (4) tends to a periodic orbit. In Fig. 2 we have drawn the "fundamental" curve $x = F(y)$ and a trajectory starting at some point P.

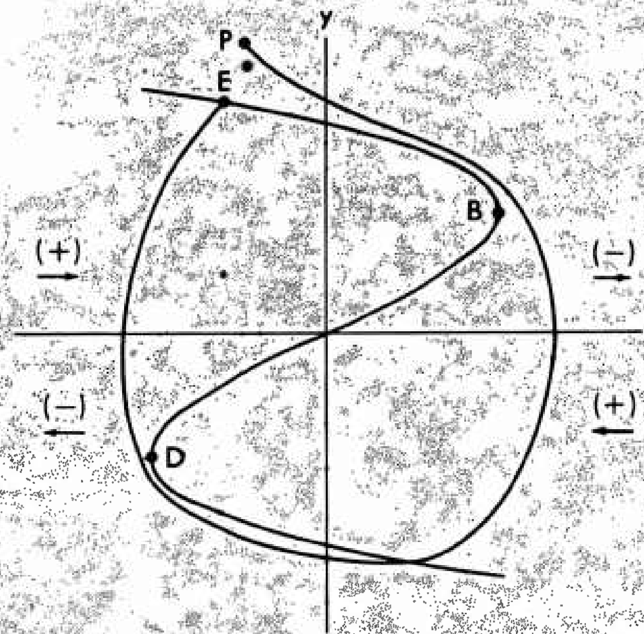

FIGURE 2

Note first that $dy/dx > 0$ at all points to the left of the curve $x = F(y)$ and such that $y > 0$ and at all points to the right of $x = F(y)$ and such that $y < 0$. We have indicated the parts of the plane where $dy/dx > 0$ or $dy/dx < 0$ by the signs $+$ and $-$, respectively. Note also that since $dx = ydt$, the value of x on any trajectory must increase in the upper half plane and decrease in the lower half plane. We have indicated this fact by the horizontal arrows.

Consider a trajectory starting at a point P at some distance from the curve $x = F(y)$. From (4) because ε is small, the value of dy/dx

must be large and negative. Thus the trajectory must drop very rapidly towards the fundamental curve $x = F(y)$. The trajectory cannot cross the fundamental curve for, if it did, the trajectory would be in a region where its slope is positive, the value of x is increasing (watch the arrow!) and the value of y is decreasing. The trajectory continues close to the fundamental curve until it passes the maximum point B and then, as can be seen from (4), dy/dx becomes large again and the trajectory drops rapidly to point C. At C the trajectory crosses the fundamental curve and dy/dx becomes negative. The trajectory must stay close to the fundamental curve for, otherwise, dy/dx becomes very large and the trajectory is forced back. Note also that the trajectory cannot cross the fundamental curve between C and D because above CD the trajectory must go toward the x-axis with a positive slope and with the value of x decreasing, conditions which are impossible to satisfy. The conclusion is then that the trajectory is close to the fundamental curve along CD and then rises very quickly to some point E on the fundamental curve. After E the trajectory keeps going around in essentially the same orbit. A rigorous proof that a periodic trajectory is approached can be found in Stokes, "Nonlinear Vibrations", Appendix IV.

The above discussion has shown that the periodic trajectory can be approximated by two arcs AB and CD of the characteristic curve and two vertical lines BC and DA as indicated in Fig. 3. To this approximation we may find the period T. On AB and DC we have

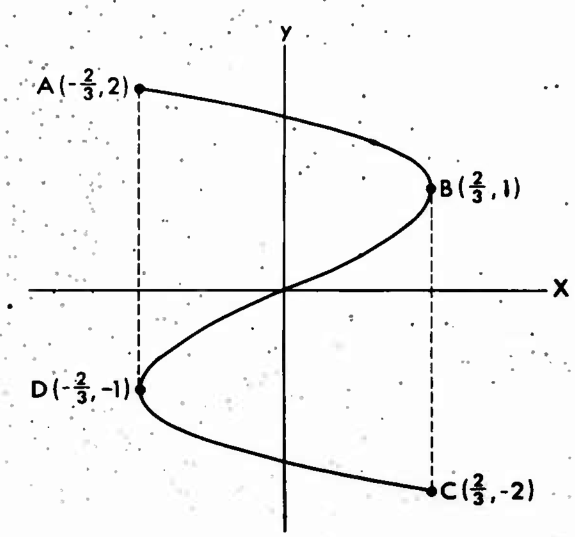

FIGURE 3

$x = F(y)$ but on BC and AD we have x constant. Since $dt = dx/y$, we have

$$T = \int \frac{dx}{y} = 2\int_A^B \frac{dF(y)}{y} = 2\int_2^1 (1 - y^2)\frac{dy}{y} = 2(\ln y - \frac{y^2}{2})\Big|_2^1 \tag{6}$$

$$= 3 - 2\ln 2 .$$

This naive attempt surprisingly gives the correct first order term for the period. We shall see, however, that to get a correction term requires a good deal of analytic manipulation. Let us suppose* that, as indicated in Fig. 4, we have a periodic trajectory which begins at some point A_1 on the fundamental curve and goes through the half cycle indicated by the points $A_1, A_2, \ldots, A_7$. The coordinates of A_i will be denoted by (x_i, y_i), $1 \leq i \leq 7$. We specify $x_2 = 2/3$, thus A_2 is

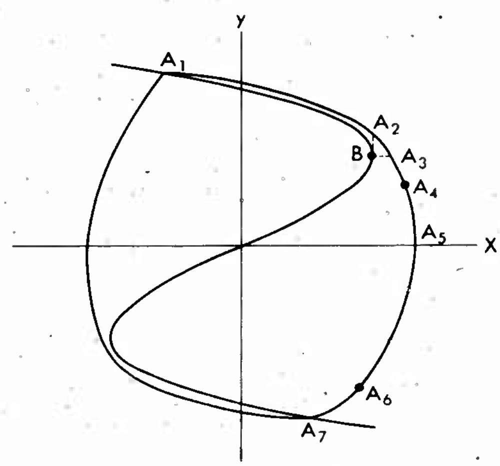

FIGURE 4

vertically above B and $y_3 = 1$, thus A_3 is on the same horizontal line as B. The other points will be specified later.

* The method used here follows very closely a paper by W. Wasow in "Procedures of Symposium on Non-Linear Circuit Theory," p. 87-97.

We begin by estimating the deviation between the curve $A_1A_2A_3$ and the curve A_1B. Put

$$u = x - F(y),$$

then, using (4) and (5), we find

$$\frac{du}{dy} = \frac{dx}{dy} - F'(y) = -\frac{\varepsilon y}{u} - F'(y)$$

$$(7) \qquad = \frac{y^2 - 1}{u}\left(u - \frac{\varepsilon y}{y^2 - 1}\right).$$

On the arc $A_1A_2A_3$, $y \geq 1$ and $u \geq 0$. Since $u_1 = 0$ and $du/dy < 0$ at A, and since along $A_1A_2A_3$ we have $dy < 0$, then $du > 0$ near A_1. If du stays positive in going from A_1 to A_3, then we must have in (7)

$$(8) \qquad u < \frac{\varepsilon y}{y^2 - 1} < \frac{\varepsilon}{y - 1}.$$

If $du = 0$ at some point y^*, then $u(y)$ has a maximum there and we have for $y < y^*$

$$u(y) \leq u(y^*) = \frac{\varepsilon y^*}{y^{*2} - 1} \leq \frac{\varepsilon}{y^* - 1} < \frac{\varepsilon}{y - 1};$$

consequently (8) is valid for all points on $A_1A_2A_3$.

The estimate (8) shows that the deviation from the arc A_1B is of the order ε as long as the trajectory is not too near B. To investigate the behavior of the trajectory in the neighborhood of B, we put

$$(9) \qquad y = 1 + \varepsilon^{1/3}v, \quad x = 2/3 + \varepsilon^{2/3}\tau$$

in (4). We find

(10) $$\frac{dv}{d\tau} = -\frac{v^2 + \frac{1}{3}\varepsilon^{1/3}v^3 + \tau}{1 + \varepsilon^{1/3}v}.$$

As ε approaches zero, this equation may be approximated by

(11) $$\frac{dv}{d\tau} + v^2 + \tau = 0.$$

Equation (11) is a Riccati equation and with the help of the standard transformation

$$v = \frac{w'}{w}$$

it may be transformed to the linear second order equation

(12) $$w'' + \tau w = 0.$$

The solution of (12) is

$$w = \alpha A_i(-\tau) + \beta B_i(-\tau)$$

where $A_i(\tau)$ and $B_i(\tau)$ are Airy functions. (For the definitions and properties of these functions see Jeffreys, "Methods of Mathematical Physics".)

The appropriate solution of (11) for our problem is the one that "matches" with the solution of (4) as $v \longrightarrow \infty$ but $y - 1 \longrightarrow 0$. Let us take $v = \varepsilon^{-r}$ where $0 < r < 1/3$. Then $y - 1 = \varepsilon^{1/3 - r} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. From (8) and (9) we find that

$$0 \leq u = \varepsilon^{2/3}(v^2 + \tau + \varepsilon^{1/3}v/3) \leq \varepsilon^{2/3}v^{-1}.$$

This simplifies to

$$- \varepsilon^{1/3} v/3 < v^2 + \tau < v^{-1}$$

and for $v = \varepsilon^{-r}$, we see that $v^2 + \tau$ mist approach zero as v approaches infinity; consequently, v is approximately equal to $(-\tau)^{1/2}$ for large values of v. Using the general solution (12), we have that

$$v = - \frac{\alpha A_i'(-\tau) + \beta B_i'(-\tau)}{\alpha A_i(-\tau) + \beta B_i(-\tau)}$$

and by studying the asymptotic behavior of $A_i(-\tau)$ and $B_i(-\tau)$ we conclude that if $v^2 + \tau = 0$, we must have

$$v = - \frac{A_i'(-\tau)}{A_i(-\tau)} \tag{13}$$

Formula (13) enables us to complete the specification of the coordinates of $A_2A_3A_4$. The point A_2 was defined by $x_2 = 2/3$ or $\tau_2 = 0$; therefore,

$$y_2 = 1 - \varepsilon^{1/3} \frac{A_i'(0)}{A_i(0)} .$$

The point A_3 was defined by $y_3 = 1$ or $v = 0$. If τ_3 is the value of τ such that $A_i'(-\tau_3) = 0$, then

$$x_3 = 2/3 + \varepsilon^{2/3}\tau_3 .$$

The point A_4 has not been defined as yet. Let us take $y_4 = 1 - \varepsilon^r$ where $0 < r < 1/3$; then $v = \varepsilon^{r-1/3} \longrightarrow \infty$ as $\varepsilon \longrightarrow 0$. Let τ_4 be the smallest value of τ such that $A_i(-\tau_4) = 0$. (The value of τ_4 is approximately 2.33.) Then from (13) we see that

(14) $$x_4 = 2/3 + \varepsilon^{2/3}\tau_4$$

in order to ensure that v_4 approach infinity.

The point A_5 is defined as the intersection of the trajectory with the x-axis; consequently, $y_5 = 0$. To find x_5, we use

$$x_4 - x_5 = \int_{x_5}^{x_4} dx = -\int_0^{y_4} \frac{\varepsilon dy}{u}$$

from (4). We may express u as follows:

$$u = (y-1)^2 + \frac{1}{3}(y-1)^3 + x - \frac{2}{3} .$$

Since on the arc A_4A_5 the value of x is always increasing (because $dx = ydt > 0$) and the value of y is always decreasing, we have $x - 2/3 > 0$ and $|y-1| \le 1$; therefore,

$$u \ge (y-1)^2[1 + \tfrac{1}{3}(y-1)] \ge \tfrac{2}{3}(y-1)^2 .$$

Using this result, we find that

$$\int_0^{1-\varepsilon^r} \frac{ydy}{u} < \frac{3}{2}\int_0^{1-\varepsilon^r} (y-1)^{-2}dy = O(\varepsilon^{-r});$$

therefore,

$$x_4 - x_5 = O(\varepsilon^{1-r}) = o(\varepsilon^{2/3}).$$

The point A_6 is defined by taking $y_6 = y_7 + \varepsilon$. Again, we use (4) for dx to get

$$x_5 - x_6 = -\varepsilon\int_{y_7+\varepsilon}^{0} \frac{ydy}{u} .$$

Noting that $x_7 = F(y_7)$ and expressing u in terms of coordinates relative to the point A_7, we have

$$
\begin{aligned}
u &= x - x_7 + \tfrac{1}{3}(y - y_7)^3 + y_7(y - y_7)^2 + (y_7^2 - 1)(y - y_7) \\
&> (y - y_7)[\tfrac{1}{3}(y - y_7)^2 + y_7(y - y_7) + (y_7^2 - 1)]
\end{aligned} \tag{15}
$$

because $x - x_7$ is positive in the lower half plane. The bracket in (15) has its minimum value when

$$y - y_7 = -3y_7/2,$$

that is, $y = -y_7/2$. Since this value of y is positive, we conclude that the minimum value of the bracket along the arc A_6A_5 will be at $y = 0$. We have then that

$$u > (y - y_7)[\tfrac{1}{3}y_7^2 - y_7^2 + y_7^2 - 1] > \tfrac{1}{3}(y - y_7).$$

Using this estimate, we get

$$|x_6 - x_5| < \varepsilon |y_7| \int_{y_7+\varepsilon}^{0} \frac{3dy}{y - y_7} = O(\varepsilon \ln \varepsilon).$$

To estimate $x_6 - x_7$ we need another estimate for u. From (7) we have

$$u\frac{du}{dy} = u(y^2 - 1) - \varepsilon y \geq -\varepsilon y$$

since on A_6A_7 it is clear that $y^2 > 1$, $u > 0$ and $-y \leq -y_6$. Integrating this inequality, we find

$$u^2 \geq -2\varepsilon y_6(y - y_7) \tag{16}$$

because $u = 0$ for $y = y_7$. With the help of (16), we have

$$|x_6 - x_7| < \varepsilon |y_7| \int_{y_7}^{y_7+\varepsilon} \frac{dy}{u} = O(\varepsilon^{1/2}) \int_{y_7}^{y_7+\varepsilon} (y - y_7)^{-1/2} dy$$

$$= O(\varepsilon).$$

Finally, combining the previous estimates for the difference between the values of successive x's, we get

$$\begin{aligned} x_7 &= x_7 - x_6 + x_6 - x_5 + x_5 - x_4 + x_4 \\ &= x_4 + O(\varepsilon^{1-r}) = 2/3 + \varepsilon^{2/3}\tau_4 + o(\varepsilon^{2/3}) \end{aligned} \tag{17}$$

by (14). Since $x_7 = F(y_7)$, we see that

$$\frac{1}{3}(y_7 - 1)^2(y_7 + 2) = -\varepsilon^{2/3}\tau_4,$$

or

$$y_7 + 2 = -\frac{3\varepsilon^{2/3}\tau_4}{(y_7 - 1)^2};$$

therefore,

$$y_7 = -2 - \frac{\varepsilon^{2/3}\tau_4}{3} + o(\varepsilon^{2/3}). \tag{18}$$

Because the trajectory is symmetric with respect to the origin, we conclude that

$$x_1 = \frac{2}{3} + \varepsilon^{2/3}\tau_4, \qquad y_1 = 2 + \varepsilon^{2/3}\tau_4/3. \tag{19}$$

Now that the coordinates of the points $A_1, A_2, \ldots, A_7$ have been determined, we can find the period. Let T_{ij} denote the time to travel between points A_i and A_j. From $dt = dx/y$, we get

$$(20)\qquad T_{12} = \int_{x_1}^{x_2} \frac{dx}{y} = + \int_{y_1}^{y_2} \frac{dF(y)}{y} + \int_{y_1}^{y_2} \frac{du}{y}.$$

But

$$(21)\qquad \int_{y_1}^{y_2} \frac{du}{y} = \frac{u_2}{y_2} + \int_{y_1}^{y_2} \frac{udy}{y^2}$$

and

$$u_2 = \frac{2}{3} - F(y_2) = F(y_3) - F(y_2) = \int_{y_2}^{y_3} dF(y) = O(\varepsilon^{2/3})$$

since $y_3 - y_2 = O(\varepsilon^{1/3})$. We have

$$(22)\qquad \begin{aligned} \frac{u_2}{y_2} &= [1 + O(\varepsilon^{1/3})] \int_{y_2}^{y_3} dF(y) = \int_{y_2}^{y_3} dF(y) + O(\varepsilon) \\ &= \int_{y_2}^{y_3} \frac{dF(y)}{y} + \int_{y_2}^{y_3} \left(\frac{1-y}{y}\right) dF(y) + O(\varepsilon) = \int_{y_2}^{y_3} \frac{dF(y)}{y} + O(\varepsilon). \end{aligned}$$

Also, with the use of (8), we see that

$$(23)\qquad \int_{y_1}^{y_2} \frac{udy}{y^2} < \varepsilon \int_{y_1}^{y_2} \frac{dy}{y-1} = \varepsilon \ln (y-1)\Big|_{y_1}^{y_2} = O(\varepsilon \ln \varepsilon).$$

Combining (20), (21), (22) and (23), we get

$$(24)\qquad T_{12} = \int_{y_1}^{y_3} \frac{dF(y)}{y} + O(\varepsilon^{1-r}).$$

Next, we evaluate

$$(25)\qquad \begin{aligned} T_{24} &= \int_{x_2}^{x_4} \frac{dx}{y} = [1 + O(\varepsilon^r)](x_4 - x_2) \\ &= \varepsilon^{2/3}\tau_4 + O(\varepsilon^{2/3 + r}). \end{aligned}$$

In the same way, just as we evaluated $x_4 - x_5$, $x_5 - x_6$ and $x_6 - x_7$, we can show that

$$T_{47} = -\varepsilon \int_{y_4}^{y_7} \frac{dy}{u} = O(\varepsilon^{1-r}) + O(\varepsilon \ln \varepsilon) + O(\varepsilon)$$

(26)

$$= o(\varepsilon^{2/3}).$$

Evaluating (24) and combining with (25) and (26), we get

$$T_{17} = \frac{3}{2} - \ln 2 + \frac{1}{2}\varepsilon^{2/3}\tau_4 + \varepsilon^{2/3}\tau_4 + o(\varepsilon^{2/3})$$

$$= .8 + 3.5\varepsilon^{2/3} + o(\varepsilon^{2/3}).$$

Since $T = 2T_{17}$, we obtain finally

$$T = 1.6 + 7.0\varepsilon^{2/3}.$$